

Disposition Polynomials and Plane Trees

William Y. C. Chen¹ and Janet F.F. Peng²

^{1,2}Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

Email: ¹chen@nankai.edu.cn, ²janet@mail.nankai.edu.cn

Abstract. We define the disposition polynomial $R_m(x_1, x_2, \dots, x_n)$ as $\prod_{k=0}^{m-1} (x_1 + x_2 + \dots + x_n + k)$. When $m = n - 1$, this polynomial becomes the generating function of plane trees with respect to certain statistics as given by Guo and Zeng. When $x_i = 1$ for $1 \leq i \leq n$, $R_m(x_1, x_2, \dots, x_n)$ reduces to the rising factorial $n(n+1) \cdots (n+m-1)$. Guo and Zeng asked the question of finding a combinatorial proof of the formula for the generating function of plane trees with respect to the number of younger children and the number of elder children. We find a combinatorial interpretation of the disposition polynomials in terms of the number of right-to-left minima of each linear order in a disposition. Then we establish a bijection between plane trees on n vertices and dispositions from $\{1, 2, \dots, n-1\}$ to $\{1, 2, \dots, n\}$ in the spirit of the Prüfer correspondence. It gives an answer to the question of Guo and Zeng, and it also provides an answer to another question of Guo and Zeng concerning an identity on the plane tree expansion of a polynomial introduced by Gessel and Seo.

Keywords: disposition, plane tree, bijection

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1 Introduction

The notation of dispositions was introduced by Mullin and Rota [4], see also, Joni, Rota and Sagan [3]. Let $(x)^n$ denote the rising factorial $x(x+1) \cdots (x+n-1)$. Assume that x is a nonnegative integer. Then $(x)^n$ can be interpreted as the number of dispositions from $[n] = \{1, 2, \dots, n\}$ to $[x] = \{1, 2, \dots, x\}$, where a disposition from $[n]$ to $[x]$ is a function from $[n]$ to $[x]$ in which the pre-images of each $i \in [x]$ are endowed with a linear order. In other words, a disposition from $[n]$ to $[x]$ can be viewed as a decomposition of a permutation of $[n]$ into x parts.

In this paper, we introduce the disposition polynomials $R_m(x_1, x_2, \dots, x_n)$ as a multivariate extension of the rising factorials by considering the number of right-to-left minima of each linear order in a disposition from $[m]$ to $[n]$. More precisely, the disposition polynomials are defined by

$$R_m(x_1, x_2, \dots, x_n) = \prod_{k=0}^{m-1} (x_1 + x_2 + \dots + x_n + k). \quad (1.1)$$

As will be seen, for the disposition polynomials $R_m(x_1, x_2, \dots, x_n)$, the exponent of x_i records the number of right-to-left minima of the i -th linear order in a disposition. For the purpose of this paper, we shall use the homogeneous disposition polynomials as given by

$$Q_m(x_1, x_2, \dots, x_n, t) = \prod_{k=0}^{m-1} (x_1 + x_2 + \dots + x_n + kt). \quad (1.2)$$

Note that the homogenous disposition polynomials have essentially the same combinatorial interpretation as that of the disposition polynomials.

In fact, we are led to the above definition of the disposition polynomials by the special case $m = n - 1$ given by Guo and Zeng [2] for the enumeration of plane trees.

Let \mathcal{P}_n denote the set of plane trees on $[n]$, where a plane tree on $[n]$ is a labeled rooted tree on $[n]$ in which the children of each vertex are linearly ordered, and let $\mathcal{P}_n^{(r)}$ denote the set of plane trees on $[n]$ with root r . For $T \in \mathcal{P}_n$, let i be a vertex of T and j be a child of i . If the smallest descendent of j is smaller than those of its brothers on the right, then j is called a younger child of i . Otherwise, j is called an elder child of i , that is, the smallest descendent of j is bigger than those of a brother on the right. Denote by $\text{young}_T(i)$ the number of younger children of i in T , and denote by $\text{eld}(T)$ the total number of elder vertices in T . Guo and Zeng [2] obtained the following formulas

$$\sum_{T \in \mathcal{P}_n} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = \prod_{k=0}^{n-2} (x_1 + x_2 + \dots + x_n + kt), \quad (1.3)$$

and

$$\sum_{T \in \mathcal{P}_n^{(r)}} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = x_r \prod_{k=1}^{n-2} (x_1 + x_2 + \dots + x_n + kt). \quad (1.4)$$

Guo and Zeng proved the above formulas (1.3) and (1.4) by induction and asked for combinatorial proofs. In answer to the questions of Guo and Zeng, we first give a combinatorial interpretation of the disposition polynomials. Then, for the case $m = n - 1$, we establish a Prüfer type correspondence between plane trees and dispositions, which implies combinatorial interpretations of both relations (1.3) and (1.4).

Replacing n by $n + 1$, t by $t - z$ and setting $r = 1$, $x_1 = x$ and $x_i = z$ for $2 \leq i \leq n + 1$, the right hand side of (1.4) becomes the polynomial

$$x \prod_{k=1}^{n-1} (x + (n - k)z + kt),$$

which is the polynomial $P_n(t, z, x)$ introduced by Gessel and Seo [1] for the enumeration of labeled trees by the number of proper vertices. Several expansions of the polynomial $P_n(t, z, x)$ have been given by Gessel and Seo [1] in terms of rooted trees with proper vertices, k -ary trees with proper vertices, k -colored ordered forests with proper vertices

and parking functions with lucky cars by using generating functions. Combinatorial proofs of some of these relations have been found by Seo and Shin [5] and by Shin [6].

With the above substitutions, (1.4) reduces to the relation

$$\sum_{T \in \mathcal{P}_{n+1}^{(1)}} x^{\text{young}_T(1)} (t - z)^{\text{eld}(T)} z^{n - \text{young}_T(1) - \text{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n - k)z + kt). \quad (1.5)$$

Guo and Zeng [2] deduced the above formula as another combinatorial interpretation of the polynomial $P_n(t, z, x)$ of Gessel and Seo, and they raised the question of finding a combinatorial interpretation of (1.5).

Our correspondence between plane trees and dispositions can be directly applied to give a combinatorial interpretation of (1.5). Indeed, the above relation holds for plane trees with any given root r , that is,

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\text{young}_T(r)} (t - z)^{\text{eld}(T)} z^{n - \text{young}_T(r) - \text{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n - k)z + kt). \quad (1.6)$$

This paper is organized as follows. In Section 2, we give a combinatorial explanation of the disposition polynomials. Section 3 provides a Prüfer type correspondence between plane trees and dispositions which leads to combinatorial interpretations of (1.3) and (1.4). Section 4 is devoted to the combinatorial proof of (1.6).

2 The generating function of dispositions

In this section, we give a combinatorial interpretation of the disposition polynomials

$$R_m(x_1, x_2, \dots, x_n) = \prod_{k=0}^{m-1} (x_1 + x_2 + \dots + x_n + k).$$

The notion of dispositions was introduced by Mullin and Rota [4] as a combinatorial explanation of the rising factorials $(x)^n = x(x + 1) \cdots (x + n - 1)$, see also Joni, Rota and Sagan [3].

Recall that a *disposition* is a function from $[m]$ to $[n]$ together with a linear order on the pre-images of each $i \in [n]$. Intuitively, a disposition can be visualized as a way of placing m distinguished balls into n distinguished boxes, where the balls in each box are linearly ordered, or equivalently, we may consider a disposition as a decomposition of a permutation on $[m]$ into n segments, where we allow a segment to be empty. We denote by $\mathcal{D}_{m,n}$ the set of all dispositions from $[m]$ to $[n]$.

For example, Figure 1 gives a disposition from $[9]$ to $[8]$.

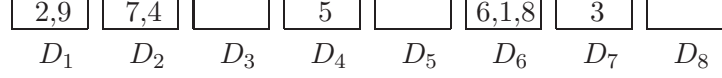


Figure 1: An example of a disposition.

Let D be a disposition from $[m]$ to $[n]$. We may write D as (D_1, D_2, \dots, D_n) , where $D_1 D_2 \dots D_n$ is a permutation of $[m]$. Recall that, for a permutation $\pi = \pi_1 \pi_2 \dots \pi_k$ of k elements, π_i is said to be a *right-to-left minimum* if $\pi_i < \pi_j$ for each $j > i$. We denote by $\text{RLmin}(D_i)$ the number of right-to-left minima in D_i . For the disposition in Figure 1, we have $\text{RLmin}(D_1) = 2$, $\text{RLmin}(D_2) = 1$, $\text{RLmin}(D_3) = 0$, $\text{RLmin}(D_4) = 1$, $\text{RLmin}(D_5) = 0$, $\text{RLmin}(D_6) = 2$, $\text{RLmin}(D_7) = 1$, $\text{RLmin}(D_8) = 0$.

As will be seen, the disposition polynomials are the generating functions of dispositions with respect to the statistics $\text{RLmin}(D_i)$. The proof of the following theorem is essentially the same argument for the combinatorial interpretation of the rising factorials.

Theorem 2.1 *For $n \geq 1$, we have*

$$\sum_{D \in \mathcal{D}_{m,n}} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)} = \prod_{k=0}^{m-1} (x_1 + x_2 + \dots + x_n + k). \quad (2.1)$$

Proof. We use induction on m . For $m = 1$, the assertion is clear. Assume that (2.1) holds for $m - 1$, that is,

$$\sum_{D \in \mathcal{D}_{m-1,n}} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)} = \prod_{k=0}^{m-2} (x_1 + x_2 + \dots + x_n + k). \quad (2.2)$$

We proceed to show that the theorem holds for m . A disposition from $[m]$ to $[n]$ can be obtained by inserting the element m in a segment of a disposition from $[m - 1]$ to $[n]$. Let (D_1, D_2, \dots, D_n) be a disposition from $[m - 1]$ to $[n]$. Write $D_i = a_1 a_2 \dots a_{r_i}$. There are $r_i + 1$ possible positions for the insertion of m in D_i . We consider two cases. Case 1. The element m is attached to the end of D_i . Let $D'_i = a_1 a_2 \dots a_{r_i} m$. It is clear that D'_i has one more right-to-left minima than D_i , that is,

$$\text{RLmin}(D'_i) = \text{RLmin}(D_i) + 1.$$

Case 2. The element m is inserted before an element in D_i . Let $D'_i = a_1 a_2 \dots a_{t-1} m a_t \dots a_{r_i}$, for $1 \leq t \leq r_i$. In this case, we have

$$\text{RLmin}(D'_i) = \text{RLmin}(D_i).$$

Since $r_1 + r_2 + \cdots + r_n = m - 1$, considering all possible insertions of m into (D_1, D_2, \dots, D_n) , we obtain that

$$\begin{aligned} \sum_{D \in \mathcal{D}_{m,n}} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)} &= (x_1 + r_1 + \cdots + x_n + r_n) \sum_{D \in \mathcal{D}_{m-1,n}} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)} \\ &= (x_1 + x_2 + \cdots + x_n + m - 1) \sum_{D \in \mathcal{D}_{m-1,n}} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)}. \end{aligned}$$

Thus, by the induction hypothesis, we find that the theorem holds for m . This completes the proof. \blacksquare

In fact, one can use the combinatorial interpretation of the coefficients of the rising factorials and the fundamental bijection for permutations to deduce the above explanation of the disposition polynomials. Recall that the coefficient of x^k in $(x)^m = x(x+1)\cdots(x+m-1)$ is the number of permutations of $[m]$ with k cycles, see Stanley [7, 1.3.4 Proposition]. The fundamental bijection is also called the standard representation of a permutation, or the first fundamental transformation. For the purpose of this paper, the standard representation of a permutation is defined as follows. Based on the cycle decomposition, we write each cycle by putting the minimum element at the end, and we arrange the cycles in the increasing order of their minimum elements. Then we erase all the parentheses.

Consider the set $\mathcal{S}_{m,n}$ of cycle representations of permutations on $[m]$ with each cycle colored by one of the n colors, say $1, 2, \dots, n$. For $\pi \in \mathcal{S}_{m,n}$, let $c_i(\pi)$ denote the number of cycles of π with color i .

Theorem 2.2 *For $n \geq 1$, we have*

$$\sum_{\pi \in \mathcal{S}_{m,n}} \prod_{i=1}^n x_i^{c_i(\pi)} = \prod_{k=0}^{m-1} (x_1 + x_2 + \cdots + x_n + k). \quad (2.3)$$

It can be seen that Theorem 2.1 can be deduced from Theorem 2.2 through the correspondence between permutations with colored cycles and dispositions. For any $\pi \in \mathcal{S}_{m,n}$, one may construct a disposition (D_1, D_2, \dots, D_n) by the fundamental bijection, where D_i is obtained from the cycles of π with color i . Clearly, we have

$$\text{RLmin}(D_i) = c_i(\pi).$$

Hence, Theorem 2.1 can be deduced from Theorem 2.2.

We define the homogenous disposition polynomials as follows

$$Q_m(x_1, x_2, \dots, x_n, t) = \prod_{k=1}^{m-1} (x_1 + x_2 + \cdots + x_n + kt).$$

For $n = m - 1$, the homogenous disposition polynomials have been used by Guo and Zeng [2]. Given a permutation $\pi = \pi_1\pi_2 \cdots \pi_m$, Guo and Zeng defined a general descent as an index i such that $\pi_i > \pi_j$ for some $j > i$. Let $\text{gdes}(\pi)$ denote the number of general descents of π . For a disposition $D = (D_1, D_2, \dots, D_n)$ from $[m]$ to $[n]$, let $\text{gdes}(D)$ denote the total number of general descents of D_i for $1 \leq i \leq n$. It is easily checked that

$$\text{gdes}(D) = m - \sum_{i=1}^n \text{RLmin}(D_i).$$

Then the homogeneous disposition polynomials have the following combinatorial interpretation

$$Q_m(x_1, x_2, \dots, x_n, t) = \sum_{D \in \mathcal{D}_{m,n}} t^{\text{gdes}(D)} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)}. \quad (2.4)$$

3 A bijection between plane trees and dispositions

In this section, we present a bijection between plane trees and dispositions in the spirit of the Prüfer correspondence, which leads to a combinatorial interpretation of the following formula of Guo and Zeng,

$$\sum_{T \in \mathcal{P}_n} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = \prod_{k=0}^{n-2} (x_1 + x_2 + \cdots + x_n + kt),$$

where \mathcal{P}_n denotes the set of plane trees on $[n]$, $\text{eld}(T)$ denotes the number of elder vertices of T and $\text{young}_T(i)$ denotes the number of younger children of vertex i of T .

We now recall some terminology. Given two vertices i and j of a plane tree T , we say that j is a *descendant* of i if i lies on the unique path from the root to j . In particular, each vertex is a descendant of itself. Denote by $\beta_T(i)$ the smallest descendant of i . A child of i means a descent j such that (i, j) is an edge of T . A vertex i is called the father of a vertex j if j is a child of i . The vertices with the same father are called brothers of each other. A vertex j of a plane tree T is called an elder vertex if j has a brother k to its right such that $\beta_T(k) < \beta_T(j)$, otherwise j is called a younger vertex. Denote by $\text{eld}_T(v)$ the number of elder children of v in T , and denote by $\text{young}_T(v)$ the number of younger children of v in T . It is not difficult to see that $\text{young}_T(v)$ equals the number of right-to-left minima of the sequence $\{\beta_T(v_1), \beta_T(v_2), \dots, \beta_T(v_m)\}$, where v_1, v_2, \dots, v_m are the children of v in linear order. Moreover, we denote by $\text{eld}(T)$ the total number of elder vertices of T and denote by $\text{young}(T)$ the total number of younger vertices of T .

For example, in Figure 2, each younger vertex is represented by a square, whereas each elder vertex is represented by a solid dot.

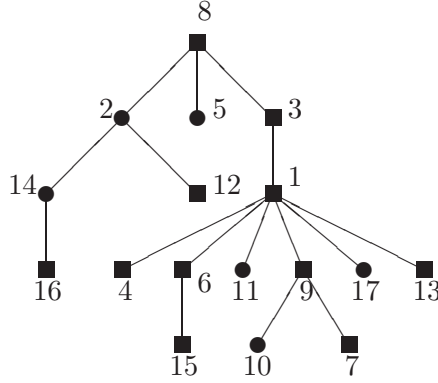


Figure 2: A plane tree $T \in \mathcal{P}_{17}$.

Theorem 3.1 *There is a bijection φ between plane trees on $[n]$ and dispositions from $[n-1]$ to $[n]$. Let T be a plane tree in \mathcal{P}_n , and let $D = (D_1, D_2, \dots, D_n)$ be the corresponding disposition under the bijection φ . Then we have $\text{young}_T(i) = \text{RLmin}(D_i)$ for all i .*

Proof. We first give a description of the map φ from \mathcal{P}_n to $\mathcal{D}_{n-1,n}$. Let T be a plane tree in \mathcal{P}_n . We proceed to construct a disposition $D = (D_1, D_2, \dots, D_n)$ through the following procedure.

First, we mark the vertices of T according to the Prüfer correspondence. More precisely, we mark the vertices of T by the numbers $0, 1, 2, \dots, n-1$. As the first step, we find the maximum leaf of T , and mark it by $n-1$. Then we remove the maximum leaf and repeat the this procedure until the root is marked by 0. These marks are called the Prüfer marks of T , which represent the order that the vertices are removed in the Prüfer correspondence. For example, Figure 3 gives the Prüfer marks of a plane tree expressed by the indices of the vertices.

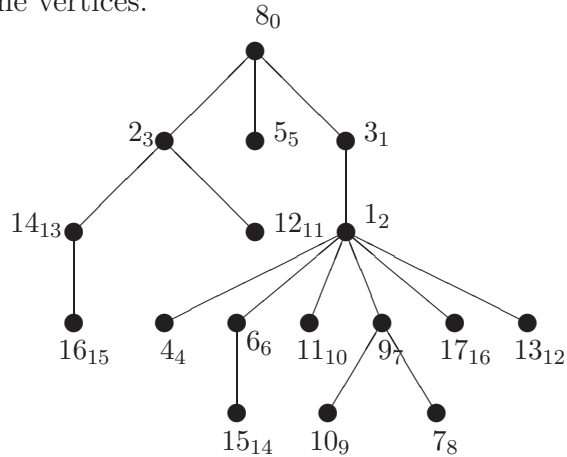


Figure 3: A plane tree with Prüfer marks $T \in \mathcal{P}_{17}$.

Using the Prüfer marks, the disposition $D = (D_1, D_2, \dots, D_n)$ can be easily constructed by setting D_i to be the set of the Prüfer marks of the children of vertex i endowed with the linear order as in T . For example, for the plane tree T in Figure 3, we have $D_1 = \{4, 6, 10, 7, 16, 12\}$, $D_2 = \{13, 11\}$, $D_3 = \{2\}$, and so on.

The above map φ is indeed a bijection. The inverse map can be described as follows. To recover a plane tree T from a disposition D , we first mark the elements of $[n]$ by $0, 1, 2, \dots, n-1$ from which one recovers the Prüfer marks of the plane tree T . We begin with the rightmost empty segment D_i , and mark the element i by $n-1$. Then we remove the empty segment D_i and the element $n-1$ from some segment of D . Repeating this procedure until the last element of $[n]$ is marked by 0.

For example, for the disposition in Figure 4, the rightmost empty segment is D_6 , thus, we mark 6 by 5. Deleting D_6 and removing 5 from D_4 , we see that D_4 becomes the rightmost empty segment. So we mark 4 by 4. Repeating this procedure, we obtain the marks $\{6_5, 4_4, 3_3, 1_2, 5_1, 2_0\}$, where the index of each element stands for its mark.

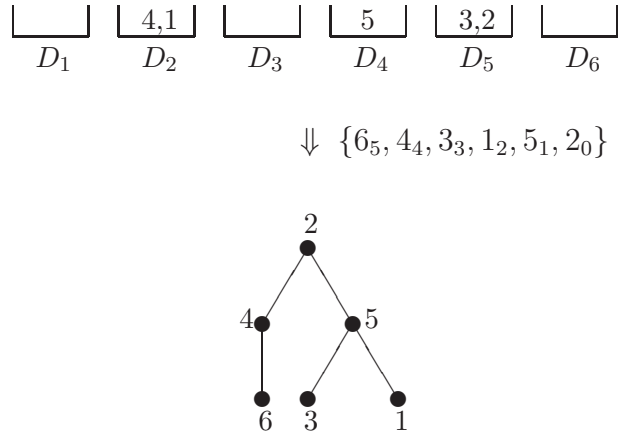


Figure 4: An example to illustrate φ^{-1} of the case $n = 6$.

Using the marks, we may construct the plane tree T by setting the root to be the element r marked by 0. If D_r is empty, then r must be 1 and T consists of the single vertex 1. Otherwise, we assume that $D_r = a_1, a_2, \dots, a_t$, and assume that b_i is marked by a_i . Set the children of r in linear order to be b_1, b_2, \dots, b_t . Repeat the above process with respect to each element b_i until we arrive at a plane tree T on $[n]$.

Take the construction of the tree in Figure 4 as an example. We have already known the marks correspondence $\{6_5, 4_4, 3_3, 1_2, 5_1, 2_0\}$. Notice that the element 2 is marked by 0, which indicates that 2 is the root of the corresponding tree. The elements in D_2 are 4, 1, which are the marks of 4, 5. Thus, the children of 2 are 4, 5 in linear order. Similarly, the element in D_4 is 5, which is the mark of the element 6, thus, the only child of 4 is

6. Continue this procedure, and we will get the corresponding tree as demonstrated by Figure 4.

Now we aim to show that the above map is indeed the inverse of φ . It suffices to prove that the marks obtained from the disposition D are the same as the Prüfer marks obtained from the plane tree T . Observe that the largest leaf l in a plane tree on $[n]$ is marked by $n - 1$. On the other hand, D_l must be the rightmost segment in the corresponding disposition, and so l is marked by $n - 1$ as well. We may repeat this argument for the element marked by $n - 2$, if there is any segment left in the disposition. Hence we reach the conclusion that we get the same marks from the disposition D and from the plane tree T .

Next we verify the relation

$$\text{young}_T(i) = \text{RLmin}(D_i),$$

where D is the corresponding disposition under φ . It is not difficult to see that the degree of vertex i of T equals the number of elements of D_i in the disposition $\varphi(T)$. Moreover, let $D_i = b_1 b_2 \cdots b_m$ and let v_1, v_2, \dots, v_m be the children of i of T in linear order. We claim that for $1 \leq j < k \leq m$, $b_j < b_k$ if and only if $\beta(v_j) < \beta(v_k)$. This property follows from the fact that the Prüfer mark of a vertex is the smallest among all its descendants. Hence we deduce that the number of younger children of vertex i of T equals the number of right-to-left minima of D_i in $\varphi(T)$. This completes the proof. ■

It is clear that Theorem 3.1 gives a combinatorial interpretation of the following relation

$$\sum_{T \in \mathcal{P}_n} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = \sum_{D \in \mathcal{D}_{n-1,n}} t^{\text{gdes}(D)} \prod_{i=1}^n x_i^{\text{RLmin}(D_i)}. \quad (3.1)$$

Combining (3.1) and the combinatorial interpretation of the disposition polynomials, we obtain a combinatorial proof of the relation (1.3), that is,

$$\sum_{T \in \mathcal{P}_n} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = \prod_{k=0}^{n-2} (x_1 + x_2 + \cdots + x_n + kt).$$

Moreover, it can be seen that our correspondence can be restricted to plane trees with a specific root r . More precisely, a disposition D corresponds to a plane tree T with root r if and only if the element 1 is contained in D_r . This leads to a combinatorial interpretation of relation (1.4).

To conclude this section, we remark that the correspondence φ is also valid for labeled rooted trees. In this case, we disregard the linear order in each segment of a disposition. In other words, φ becomes a correspondence between labeled rooted trees and decompositions of $[n - 1]$ into n components. Under this correspondence, the empty sets in a decomposition correspond to leaves of a labeled rooted tree, and more generally, the cardinalities of D_i correspond to the degrees of the rooted trees.

4 The Gessel-Seo polynomials

In this section, we use the correspondence between plane trees and dispositions to give a combinatorial interpretation of the following expansion of the Gessel-Seo polynomial,

$$\sum_{T \in \mathcal{P}_{n+1}^{(1)}} x^{\text{young}_T(1)} (t - z)^{\text{eld}(T)} z^{n - \text{young}_T(1) - \text{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n - k)z + kt), \quad (4.1)$$

where $\mathcal{P}_{n+1}^{(1)}$ denotes the set of plane trees on $[n + 1]$ with root 1. Guo and Zeng [2] derived the above identity by using generating functions and asked for a combinatorial proof. In fact, as a consequence of (1.4), that is,

$$\sum_{T \in \mathcal{P}_n^{(r)}} t^{\text{eld}(T)} \prod_{i=1}^n x_i^{\text{young}_T(i)} = x_r \prod_{k=1}^{n-2} (x_1 + x_2 + \cdots + x_n + kt),$$

we find that (4.1) holds for plane trees on $[n + 1]$ with any specific root r by replacing n by $n + 1$ and setting $x_r = x$, $x_i = z$ for any $i \neq r$. As will be seen, our correspondence between plane trees and dispositions serves as a direct combinatorial interpretation of this fact.

Theorem 4.1 *For $n \geq 1$ and $1 \leq r \leq n + 1$, we have*

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\text{young}_T(r)} (t - z)^{\text{eld}(T)} z^{n - \text{young}_T(r) - \text{eld}(T)} = x \prod_{k=1}^{n-1} (x + (n - k)z + kt), \quad (4.2)$$

where $\mathcal{P}_{n+1}^{(r)}$ is the set of plane trees on $[n + 1]$ with root r .

Proof. Replacing t by $t + z$, we may rewrite (4.2) as follows,

$$\sum_{T \in \mathcal{P}_{n+1}^{(r)}} x^{\text{young}_T(r)} t^{\text{eld}(T)} z^{n - \text{young}_T(r) - \text{eld}(T)} = x \prod_{k=1}^{n-1} (x + nz + kt). \quad (4.3)$$

We first give a combinatorial interpretation of the right hand side of (4.3). By the combinatorial interpretation of the disposition polynomials, we see that the Gessel-Seo polynomial $P_n(t + z, z, x)$ is the generating function of dispositions $D = (D_1, D_2, \dots, D_{n+1})$ from $[n]$ to $[n + 1]$ with the element 1 contained in D_r , where a right-to-left minimum in D_r is given a weight x , a right-to-left minimum in D_i ($i \neq r$) is given a weight z , and any other element is given a weight t .

For a disposition $D = (D_1, D_2, \dots, D_{n+1})$ with which the element 1 appears in D_r , let T be the plane tree corresponding to D under the bijection φ in Theorem 3.1. It is

easily seen that T has root r , namely, $T \in \mathcal{P}_{n+1}^{(r)}$. Moreover, a younger child of vertex i of T corresponds to a right-to-left minimum in D_i , and an elder child of vertex i of T corresponds to an element which is not a right-to-left minimum in D_i for $1 \leq i \leq n+1$. Hence T has weight

$$x^{\text{young}_T(r)} t^{\text{eld}(T)} z^{n - \text{young}_T(r) - \text{eld}(T)},$$

as expected. This completes the proof. ■

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